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A Test Of The Chiral E_8 Current Algebra On A 6D Non-Critical String

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Abstract

Compactifying the E_8 non-critical string in 6D down to 5D the 6D strings give rise to particles and strings in 5D. Using the dual M-theory description compactified on an elliptically fibered Calabi-Yau we compare some of the 5D BPS states to what one expects from non-critical strings with an E_8 chiral current algebra. The E_8 multiplets of particle states comprise of 2-branes wrapping on irreducible curves together with bound states of several 2-branes.

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1 Introduction

It was argued in [1] that the non-critical strings related to small E_8 instantons carry a chiral E_8 current algebra. In [2, 3] this algebra was realized as fermions in the F-theory setting and in [5] the E_8 lattice was given a geometrical interpretation as the integral cohomology lattice of a 4-manifold whose collapse is responsible for the phase transition. The authors of [5] suggested a way to identify geometrically the states corresponding to this current algebra and the purpose of the present paper is to work out some of the details of their suggestion.

Compactifying the 6D theory at the phase where a non-critical string appears down to 5D there is a dual description of M-theory on a Calabi-Yau [6, 7, 3]. The BPS string states in 6D become stringy states and particle states in 5D and are identified with 2-branes wrapped on 2-cycles and a 5-brane wrapped on a 4-cycle respectively [3, 5]. From the non-critical string point of view, the states must fall into representations of E_8 and the multiplicity of particle states must correspond to the multiplicity in a chiral E_8 current algebra [5].

In the situation at hand one has to consider bound states of 2-branes wrapping on different 2-cycles that intersect [8]. Indeed, such states are needed to complete E_8 multiplets. The general techniques for studying such bound states have been developed in [9] and this particular problem of bound states of 2-branes has been studied in [8] where their existence was necessary to complete U-duality multiplets. An analogous question for 5-branes has been recently studied in [10] where, remarkably, a non-critical string in 4D appeared.

Only slight modifications are needed for our case – first there is less supersymmetry and the 2-brane motion is frozen in 2 dimensions (orthogonal to the collapsing 4-cycle) and second we are in M-theory and not type-II string theory.

Taking into account the bound states – or *reducible curves* – and using the techniques of [11, 12] and [3] for calculating the states that arise from the moduli space of such curves and their Lorenz, E_8 and other quantum numbers we can compare the geometrical computation with the expected physical result from the low-energy of the non-critical string, i.e. the E_8 chiral current algebra and transverse oscillations. This comparison is similar to the identification of the BPS states of the heterotic string with cohomologies of symmetric products of K3-s made in [11, 12].

The paper is organized as follows. Section (2) is a review of some of the facts about non-critical strings and the F-theory construction as well as the compactification to 5D. Section (3) is a geometrical calculation of the multiplicity of stringy states in 5D. Section (4) deals geometrically with some of the particle 5D states. Section (5) is devoted to 2-branes wrapped on reducible curves, or bound states of 2-branes, and finally in section (6) the geometrical results are compared to the physical expectation. Appendix (A) discusses in more detail the $H^2(\mathbf{Z})$ cohomology of the almost del Pezzo surface that shrinks at the phase transition [5]. Appendix (B) includes some algebraic details of the counting of curves.

I have recently learned of another paper which contains similar results using different arguments [23].

2 E_8 non-critical strings in F-theory

This section is a review of some of the relevant facts from [13, 4, 1, 2, 3, 5]. The purpose of this review is also to set the geometrical notation.

The 6D heterotic string vacua compactified on K3 where described in [4, 5] as F-theory on an elliptic 3-fold. In this description phase transitions at points where tensionless strings appear correspond to collapsed cycles in the 3-fold [4, 2, 3, 5]. In this paper, we will be interested specifically in vacua near the point where a small E_8 instanton collapses. In the physical language this point connects two phases [1, 2]. One phase is the finite size instanton which breaks the E_8 gauge symmetry while the other phase has an extra tensor multiplet [13] but the E_8 symmetry is restored (locally). In F-theory this phase transition corresponds to blowing up a point in the base of the elliptic fibration.

The geometrical setting that describes that phase in F-theory is as follows [3, 5].

It is sufficient to restrict to the vicinity of the blown-up point. The 3-fold on which F-theory is compactified has a 4 dimensional base B . Picking local analytic coordinates z_1, z_2 on B such that the blown-up point is at $z_1 = z_2 = 0$ – the blow-up replaces the single point at the origin with an entire \mathbf{CP}^1 such that when we approach the origin along a line with fixed $\lambda = z_1/z_2$ with $z_1, z_2 \rightarrow 0$ we land on the point corresponding to λ on \mathbf{CP}^1 . It is allowed to blow-up a point if we can extend the elliptic fibration to the \mathbf{CP}^1 (the “exceptional divisor”).

Writing the elliptic fibration as [14]:

$$y^2 = x^3 - f(z_1, z_2)x - g(z_1, z_2) \quad (1)$$

the Calabi-Yau constant holomorphic 3-form is given by

$$\Omega = dz_1 \wedge dz_2 \wedge \frac{dx}{y} \quad (2)$$

In the vicinity of $z_1 = z_2 = 0$ the good coordinates are z_2 and λ (away from $\lambda = \infty$) or z_1 and $\frac{1}{\lambda}$ (away from $\lambda = 0$). The coordinates x and y are singular, but a change of coordinates to ξ, η where:

$$x = z_2^{2n}\xi, \quad y = z_2^{3n}\eta \quad (3)$$

will be good provided that f is of degree $4n$ and g of degree $6n$. In that case we have

$$\eta^2 = \xi^3 - f_{4n}(\lambda, 1)\xi - g_{6n}(\lambda, 1) \quad (4)$$

The holomorphic 3-form becomes

$$\Omega \sim -z_2^{1-n} d\lambda \wedge dz_2 \wedge \frac{d\xi}{\eta} \quad (5)$$

so for $n = 1$ Ω is nonzero and nonsingular on the exceptional divisor. This fixes f to be homogeneous of degree 4 and g to be homogeneous of degree 6. In terms of the number of 7-branes (i.e. singular fibers) we find that the discriminant

$$\Delta_{12}(z_1, z_2) = \frac{f^3}{27} - \frac{g^2}{4} \quad (6)$$

has 12 roots in $\lambda = z_1/z_2$ on the \mathbf{CP}^1 so 12 7-branes pierce the exceptional divisor [2, 3]. Only a maximum of eight of them can be given a perturbative type-IIB description simultaneously.

The resulting surface (2 complex dimensions) given by The elliptic fibration over the exceptional divisor \mathbf{CP}^1 :

$$y^2 = x^3 - f_4(\lambda)x - g_6(\lambda) \quad (7)$$

is the surface that collapses at the tensionless string point. We will denote this surface by \mathcal{S} . Its only non-vanishing Hodge number (other than $h^{0,0}$ and $h^{2,2}$) is $h^{1,1} = 10$ and its intersection form is

$$E_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (8)$$

It was explained in [5] that the E_8 is related to the physical E_8 gauge group of the small instanton and we will explore this relation further in the next sections.

The 2-cycles are (as described in [5]) given by f, b, e_1, \dots, e_8 , where f is the fiber, b the base and e_i are mixed. A typical 2-cycle that is a combination of the e_i -s is given by picking two 7-brane points (that is, points where the fiber degenerates) λ_1 and λ_2 on \mathbf{CP}^1 and a path γ on \mathbf{CP}^1 from λ_1 to λ_2 . At λ_1 there is a unique 1-cycle of the fiber that shrinks to zero. As we go along γ this cycle grows. As we reach λ_2 a 1-cycle has to shrink again, if it is this cycle that we transported, then we have drawn a 2-cycle which is of genus zero. As this construction will be important in what follows, we have described it in more detail in appendix (A).

Physically, the 6D F-theory vacua were described [14] as type-IIB string vacua on a 4-manifold (the base B of the Calabi-Yau) with 7-branes of different types filling the entire uncompactified 6D as well as analytic 2-cycles inside the base B . When a point of B is blown-up there is an additional small 2-cycle from that point. The 3-brane of type-IIB can wrap on that 2-cycle and this is the string with small tension that corresponds to the transition [2, 3, 5].

In this paper we will concentrate on the small E_8 instanton transition. The F-theory description shows that there are other transitions in the moduli space corresponding to other 4-cycles that shrink [5]. Those (as well as the small E_8 have been recently discussed in [15].

2.1 Compactification to 5D

Compactifying the six-dimensional theory on \mathbf{S}^1 with radius R we end up at a point on the moduli space of the heterotic string on $K3 \times \mathbf{S}^1$. That theory is equivalent to M-theory on a Calabi-Yau as discussed in [16, 6, 7, 3].

We are interested in that region of moduli space that emanates from the 6D small E_8 instanton point. After compactification to 5D the E_8 heterotic string is dual to $SO(32)$ but if we do not turn on E_8 Wilson lines on \mathbf{S}^1 the T-dual radius will be $R' = \frac{R}{1+2R^2}$ [17] and will be never large. With a special value of the Wilson loop we can reach the large radius of the $SO(32)$ theory and the non-perturbative phase that emanates from the small instanton point

becomes the Coulomb phase of Witten’s field theory for the small $SO(32)$ -instanton[18], i.e. a phase where the non-perturbative $Sp(1)$ of [18] has a Wilson loop along \mathbf{S}^1 .

Turning on an E_8 Wilson loop is equivalent to turning on certain C_3 (the 3-form of M-theory) expectation values.

Finally, a further compactification to 4D brings us to the duality between type-IIA on a Calabi-Yau and the Heterotic string on $K3 \times \mathbf{T}^2$ first found in [19]. An extensive description of the various phases as seen in 4D has been recently given in [20].

2.2 Counting states in 5D

The strings of 6D can either wrap around the 6th dimension to give a particle in 5D, or be reduced to give a string state. The 6D strings have an E_8 chiral current algebra on them [1]. It thus is interesting to see how the E_8 chiral current algebra is exhibited on the stringy states and how the particle states fall into multiplets of E_8 .

We will work in the setting of M-theory compactified on the elliptically fibered Calabi-Yau.

The structure of BPS charges was described in [16, 6, 7]. There are $h^{1,1} - 1$ vector multiplets coming from Kähler deformations of the CY except for the overall volume which is in a hyper-multiplet. The vector-moduli space is completely determined by the integer intersection numbers of $(1, 1)$ cycles of the CY. In 5D BPS charges are real and are determined from the intersection numbers [6].

The M-theory low-energy description is applicable when the size of the CY is large. On the other hand the area of the fiber is $R_E^{-4/3}$ where R_E is the radius of \mathbf{S}^1 in Einstein units. Since the overall volume is in a hyper-multiplet and decouples from the vector multiplets we can extrapolate to small overall volumes.

3 Stringy states in 5D

The six-dimensional “non-critical” strings at the small E_8 instanton point have an E_8 chiral current algebra on them.

In F-theory language, the E_8 is realized as 16 left moving fermions on the world-sheet as follows [2, 3]. A D-brane analysis shows that on the intersection of a 7-brane with a 3-brane there live 2 chiral fermions. In our case there are 12 such intersection points so it would seem that we have 24 fermions. The suppression of 8 fermion zero modes out of the total 24 is similar to the suppression of 4 $U(1)$ -s out of the 24 in the case of F-theory on K3 [14]. The situation in the latter case was clarified in physical terms in [21] where it was shown that the interactions of these $U(1)$ -s with the 2-form fields in the bulk make some of the gauge fields massive. The basic reason is that the 7-branes are not of the same (p, q) type.

In our case, the suppression of 8 fermionic modes occurs because of the $U(1)$ gauge field that lives on the 3-brane that wraps on the base. This gauge field has no resulting dynamical degrees of freedom but the fermions at the intersections of the 7-branes and the 3-brane are *charged* under that $U(1)$. Because of the varying coupling constant of F-theory and because we have to use S-duality to get a local perturbative D-brane description when moving along the base, some of the fermions are magnetically charged and some are electrically charged in any global description. The surviving fermionic modes are those that correspond to a global $U(1)$ field configuration with sources at some of the 7-branes (where fermion modes are excited). Such $U(1)$ fields are linear combinations of the following configurations. Take any pair of 7-branes and a path connecting them in such a way that as we move one 7-brane along the path towards the other 7-brane we end-up with *mutually local* 7-branes (i.e. 7-branes of same (p, q) -type). Now we can take one 7-brane to be a positive source of $U(1)$ and the other to be a negative source of $U(1)$ and as we deform the 7-branes back to the original position along that path we used above we end up with a global $U(1)$ configuration that is consistent with the S-dualities that we have to perform to relate different patches of the base.

The reason for describing the $U(1)$ field configurations in that way is to make contact later on with the geometry of the base. Indeed, the above description is in 1-to-1 correspondence with the construction of the e_1, \dots, e_8 $(1, 1)$ 2-cycles in \mathcal{S} discussed briefly in the previous section and elaborated in appendix (A). In any case, the analogy shows that there are precisely 8 linearly independent such configurations corresponding to the 2×8 massless fermion modes that generate the E_8 chiral current algebra.

After compactification to 5D the connection with the geometrical description is even more manifest. The stringy states come from 5-branes of M-theory which wrap around the 4-cycle \mathcal{S} in the Calabi-Yau which is now physical. On the 5-brane there lives a tensor multiplet and the zero modes of the anti-self-dual 2-form $B_{\mu\nu}^{(-)}$ will give rise to fields on the world-sheet. The zero-modes which are proportional to the cohomology classes (dual to) e_1, \dots, e_8 become 8 left-moving chiral bosons on the string world-sheet. The bosons live on the E_8 lattice and thus correspond in a natural way to the chiral E_8 current algebra. The remaining modes of $B_{\mu\nu}^{(-)}$ and the scalars of the tensor multiplet are in 1-to-1 correspondence with transverse oscillations of the string.

3.1 A note on the geometrical interpretation of the 6D fermions

As an aside, let us give another geometrical interpretation for the fermions on the intersection of a 3-brane and a 7-brane. T-duality replaces the configuration with a 6-brane and a 2-brane of type-IIA. The reason we work with that configuration is that the 6-brane, being a source for the 1-form which corresponds to rotations in the 11th dimension (i.e. a Kaluza-Klein monopole), is completely geometrical and is described by having the \mathbf{S}^1 of the 11th dimension in a non-trivial $U(1)$ -bundle around the 6-brane.

Let the 6-brane be in $x_7 = x_8 = x_9 = 0$ and let the 2-brane be in $x_1 = \dots = x_6 = 0$, $x_9 = a$ (x_0 being time). The scalar fields that live on the 2-brane correspond to translations in the ambient directions [9]. There is also a gauge field A_μ which in 3D is dual to a scalar ϕ_{11} . This scalar corresponds to the coordinate of the 2-brane along the 11th dimension which is a circle. As we increase a from negative values to positive values, we make the 2-brane pass through the 6-brane. This passage is accompanied by a twist to the $U(1)$ -bundle which is the compactified 11th direction. Thus as the 2-brane emerges for positive a , the $U(1)$ bundle on it is twisted. This means that as a passes zero, we get an increase of 2π in:

$$\oint_C \partial_j \phi_{11} dx^j \quad (9)$$

where C is a contour that surrounds the origin in the 2-brane two-dimensional space. Now, ϕ_{11} is dual to the gauge field so:

$$\partial_j \phi_{11} = \epsilon_{jk} E_k \quad (10)$$

where $E_k = \dot{A}_k$ is the electric field-strength. We find that there is a jump in $\nabla \cdot E$ in the origin, which corresponds to a jump of one unit in the electric charge at the origin.

Let us see to what this corresponds physically after T-duality. If we wrap the x_9 direction around a circle \mathbf{S}^1 , and T-dualize we find a 7-brane in the $x_7 = x_8 = 0$ position and a 3-brane in the $x_1 = \dots = x_6 = 0$ position. They both wrap around x_9 and the variable a above transforms into the $U(1)$ Wilson loop around x_9 . The statement about changing a from negative values to positive values is the geometrical manifestation of the fact that the massless fermions that live on the $7 \perp 3$ intersection are charged. When there is a Wilson loop, the modes around the x_9 are fractional ψ_{n+a} . When a moves from a negative to a positive value, one mode from the Dirac sea becomes of positive energy, so there is an effective change of one unit of charge in the vacuum.

4 Particle states

To count particle states we need to wrap the 2-brane around analytic 2-cycles. Let us first review the quantum numbers of such states. The spin of the states was determined in [3] as follows. If \mathcal{M} is the moduli space of such analytic 2-cycles (including flat gauge connections of the 2-brane which will be elaborated below) then the states correspond to the cohomology $\bigoplus_{p,q} H^{p,q}(\mathcal{M})$ and the representation in the little group $SO(4) = SU(2)_1 \times SU(2)_2$ was determined in [3] by performing a Lefschetz decomposition of $H^*(\mathcal{M})$ into representations of $SU(2)_1$ (trivial under $SU(2)_2$) such that $J_3 = \frac{1}{2}(p + q - \dim_{\mathbf{C}} \mathcal{M})$ and tensoring with half a hyper-multiplet $2(\underline{1}, \underline{1}) \oplus (\underline{1}, \underline{2})$. In particular, all those states have $SU(2)_2$ spins at most $\frac{1}{2}$ and $SU(2)_1$ spins at most $\frac{1}{2} \dim_{\mathbf{C}} \mathcal{M}$.

From the “physical” point of view, these states come from wrapping modes of the non-critical string. These strings carry a left-moving E_8 chiral current algebra at level $k = 1$ as well as 4 free fields corresponding to oscillations in transverse directions and their right moving super-partners which are 2 copies of right moving fermions in the $\underline{2}$ of $SU(2)_1$ (see e.g. the appendix of [1]).

Scalar BPS states correspond to excitations of left-movers only [8, 11, 12].

The BPS states have charges corresponding to the momentum along the 6th direction, the

“winding number” of the non-critical string and the E_8 representation. The 6th momentum is given by $L_0 - \bar{L}_0$ of the CFT and we will soon identify it geometrically.

In this section we will describe the particle states mainly in geometrical terms. In addition to irreducible curves we will see that there are also reducible curves or *bound states* of 2-branes. It turns out that they are indeed needed to complete E_8 multiplets and to reproduce the expected results from the chiral current algebra calculation. We will discuss the reducible curves more in section (5) and perform a more detailed comparison with the non-critical string states in section (6).

4.1 Curves in \mathcal{S}

The cohomology class of a curve D in \mathcal{S} is a combination

$$D = nb + mf + \sum_{i=1}^8 l_i e_i \quad (11)$$

where in the notation of section (2), b is the section of the fibration, f is the fiber

$$b = \{x = y = \infty\}, \quad f = \{z = \text{const}\} \quad (12)$$

and e_i are the other 8 cycles with intersection form E_8 . The other non-zero intersection numbers are

$$b \cdot b = -1, \quad b \cdot f = 1 \quad (13)$$

(the first equation is deduced from (3)).

The canonical bundle $\mathcal{K}_{\mathcal{S}}$ has first Chern class

$$c_1(\mathcal{K}_{\mathcal{S}}) = -f \quad (14)$$

and the genus of the curve D is given by

$$2g - 2 = D \cdot D + D \cdot \mathcal{K}_{\mathcal{S}} = -n^2 + (2m - 1)n + \vec{l}^2. \quad (15)$$

As explained in [5], the Weyl group of E_8 acts on the divisors D . Performing a closed loop in the moduli space (of parameters of f_4 and g_6) the surface \mathcal{S} returns to itself up to an automorphism which by the Lefschetz theorem acts as a Weyl reflection on the cohomology classes e_i .

From formulas in [6] we can see that P_6 , the momentum along the 6th direction \mathbf{S}^1 , is given by the coefficient of the fiber $-m$.

$$P_6 = m. \tag{16}$$

Finally, the winding number of the non-critical string is n .

4.2 single covers

Single covers are states with $D \cdot f = n = 1$. They are given by sections of the elliptic fibration. We thus need meromorphic functions $x(\lambda)$ and $y(\lambda)$ that satisfy

$$y(\lambda)^2 = x(\lambda)^3 - f_4(\lambda)x - g_6(\lambda). \tag{17}$$

Let us first take x and y to be polynomials.

If x is a polynomial of degree $2d$ then y is of degree $3d$. The intersection $D \cdot b$ is given by the intersection with the generic section $x = y = \infty$. $x(\lambda) = \infty$ requires $\lambda = \infty$. Because of the rescaling $x(\lambda) \rightarrow x(\lambda)/\lambda^2$ in the vicinity of $\lambda = \infty$ we find $d - 2$ solutions, so that

$$m = 1 + D \cdot b = d - 1 \tag{18}$$

D is isomorphic to the base and so has genus 0, thus

$$-2m = \vec{l}^2 \tag{19}$$

Since (17) is of degree $6d$ we need to adjust $(2d + 1) + (3d + 1)$ parameters in $x(\lambda)$ and $y(\lambda)$ to solve $(6d + 1)$ equations in (17). This can generically be done only for $d = 2$. For $d = 2$ $x(\lambda)$ is quadratic and (17) implies polynomial equations on the coefficients. It can be checked that there are exactly 120 solutions for $x(\lambda)$. Taking into account the \pm sign in $y(\lambda)$ we find 240 states. They should combine, as anticipated in [5], with 8 more states to form the **248** of E_8 .

What are the additional 8 states? They must be in the cohomology class of $b + f$. There can be no smooth curves in that class because $(b + f) \cdot f = 1$ implies that it must be a section and thus have genus 0, but the genus formula gives 1 for an irreducible curve of class $b + f$. Moreover, unlike the case of K3 [12], \mathcal{S} has only one complex structure for a given

metric. Thus the $b+f$ state can only be a bound state of a 2-brane of class b (i.e. the section $x = y = \infty$) with a 2-brane wrapped around f . This can be thought of as a reducible curve $(\frac{1}{x})(z - z_0) = 0$. The fiber has a *geometrically* a moduli space of \mathbf{CP}^1 and we should find that there are 8 bound states at threshold. We will discuss this point in the next section, but let us go on to larger values of m .

States with $m = 2$ cannot come from polynomial $x(\lambda)$ -s since for $d > 2$ there are no generic solutions. By naive counting of the number of variables (i.e. coefficients in $x(\lambda)$ and $y(\lambda)$) we see that for $m = 2, 3, 4$ there can be polynomial solutions only for non-generic points in the the moduli space (coefficients of f_4 and g_6) and for $m > 4$ there are no polynomial solutions at all. We will prove this more rigorously in appendix (B).

The more general meromorphic solution is given by

$$x(\lambda) = \frac{P(\lambda)}{Q(\lambda)^2}, \quad y(\lambda) = \frac{R(\lambda)}{Q(\lambda)^3}, \quad (20)$$

where P, Q, R are polynomials in λ that satisfy

$$R(\lambda)^2 = P(\lambda)^3 - f_4(\lambda)P(\lambda)Q(\lambda)^4 - g_6(\lambda)Q(\lambda)^6 \quad (21)$$

Let

$$d_Q = \deg(Q(\lambda)), \quad d_P = \deg(P(\lambda)), \quad d_R = \deg(R(\lambda)). \quad (22)$$

The number of variables (coefficients of P, Q, R up to an overall constant) is

$$d_P + d_Q + d_R + 2 \quad (23)$$

If $d_P > 2d_Q + 2$ then from (21) $d_R = \frac{3}{2}d_P$ and the number of equations (coefficients in (21)) is $2d_R + 1 = 3d_P + 1 > d_P + d_Q + d_R + 2$ so generically there is no solution and in fact (similarly to the proof in appendix (B)), for $d_P > 2d_Q + 8$ there is never a solution. For

$$d_P = 2d_Q + 2, \quad d_R = 3d_Q + 3 \quad (24)$$

The number of variables equals the number of equations and there is a finite number of solutions which must correspond to highest weights of an E_8 representation. The momentum m of these states is given by the intersection with the base b , i.e. the number of poles

$$m = 1 + d_Q \quad (25)$$

Since the curve is still isomorphic to the base the genus formula identifies the E_8 representation as the states with lattice vector \vec{l} satisfying

$$2m = 2 + 2d_Q = \vec{l}^2 \quad (26)$$

4.3 Double covers

For $n = 2$ the curves $(x(\lambda), y(\lambda))$ intersect the fiber twice. Thus they are double-valued functions of λ . They must be of the form

$$x(\lambda) = P(\lambda) + Q(\lambda)\sqrt{R(\lambda)}, \quad y(\lambda) = S(\lambda) + T(\lambda)\sqrt{R(\lambda)}, \quad (27)$$

where P, Q, R, S, T are meromorphic functions and the same \sqrt{R} appears in both equations.

The simplest case is

$$x(\lambda) = P_2(\lambda), \quad y(\lambda) = \sqrt{P_2^3 - f_4 P_2 - g_6} \quad (28)$$

with

$$\deg(P_2) \leq 2 \quad (29)$$

This is a double cover of the base with (generically) 6 branch points. It doesn't intersect $x = y = \infty$ (because of the transformation $\xi = \frac{x}{\lambda^2}$ at the vicinity of infinity). The genus is $g = 2$. It follows that the cohomology class is

$$[D] = 2b + 2f \quad (30)$$

The *geometrical* moduli space of such curves is the coefficients in $P_2(\lambda)$. When any of the coefficients becomes infinite, we can factor the infinity out to write

$$P_2(\lambda) \sim \infty \times (\lambda - \lambda_0)(\lambda - \lambda_1) \quad (31)$$

so it becomes a reducible curve – a bound state of four curves: $\lambda - \lambda_0$, $\lambda - \lambda_1$ and twice $x = y = \infty$. We learn from this that the “direction” of infinity matters so that the moduli space of the coefficients of P_2 is actually \mathbf{CP}^3 . We note also that at additional 120 points in \mathbf{CP}^3 the curve becomes reducible into two curves of classes $b + f \pm \vec{l}$ with $\vec{l}^2 = -2$, which are precisely the curves we discussed in the previous section. This happens when $P_2^3 - f_4 P_2 - g_6$ is a perfect square. However, we believe that again those two curves are in a bound state.

On this \mathbf{CP}^3 we have to fiber the moduli space of flat $U(1)$ bundles on the $g = 2$ Riemann surface. This is exactly the same setting as in [12] and we find $H^*(\mathcal{S}^3/S_3)$. Indeed as in [12], there is a unique curve in \mathcal{S} that passes through 3 generic points in \mathcal{S} . The conditions for passing through 3 generic points translates into 3 linear equations in the coefficients of the corresponding quadratic polynomial in λ .

5 reducible curves

We have seen in the previous section that in order to complete an E_8 multiplet there must exist states corresponding to a 2-brane wrapped on a 2-cycle in the cohomology class $b + f$. Those states will join the curves in the $b + f + \vec{l}$ class with $\vec{l}^2 = -2$ and together they will form 248 states in the 248 of E_8 . We saw however, that there is no irreducible curve in the class $b + f$ so the extra required states must be *bound states* of two 2-branes one wrapped on the base b and the other wrapped on the fiber f .

The purpose of this section is to argue that such states indeed exist and their E_8 quantum numbers correspond in a natural way to the Cartan subalgebra of E_8 – that is the 8 missing states in 248.

The general framework for bound-state questions has been developed in [9, 8] (and see also [10] for a similar discussion for 5-branes).

On each of the two 2-branes there live 8 bosonic fields whose zero modes correspond to translations of the 2-brane in a transverse direction. One has to determine what low energy fields live on the intersection and how they interact with the low-energy fields on the two 2-branes. Then, one has to determine whether the system has a normalizable state (i.e. its wave-function decays fast enough as the separation between the 2-branes increases).

This is very similar to the problem studied in [8] and indeed we will use the technology developed there.

Let us start by describing the E_8 quantum numbers of such a bound state.

On a 2-brane of M-theory there lives the dimensional reduction of $\mathcal{N} = 1$ $U(1)$ SYM from 10D down to 3D, since this is what lives on a 2-brane of type-IIA [9]. The 11D Lorentz symmetry is manifest only after dualizing the gauge field to obtain 8 scalars on the world-

volume which represent transverse oscillations.

For the 2-brane that wraps around the base b , 4 out of the 8 scalars are massive, because the base is an isolated 2-cycle in the CY. For the 2-brane that wraps the fiber f , 2 out of the 8 scalars are massive because the 4-cycle \mathcal{S} is also isolated. Out of the remaining 6 scalars 2 have the moduli space which is the base \mathbf{CP}^1 corresponding to moving the fiber along the base.

Although the *dynamical* degrees of freedom are the same after the world-volume duality that replaces the gauge field with the 8th scalar, we do lose the global degrees of freedom of flat gauge connections. In our case we do have to take into account the flat gauge connections of the 2-brane that wraps around the fiber f . The resulting setting is exactly as in [12]. The moduli space of flat gauge connections on the fiber is isomorphic to the fiber itself and the entire moduli space is naturally isomorphic to \mathcal{S} . The states that we obtain correspond to the cohomology of this moduli space. Out of $h^{1,1}(\mathcal{S}) = 10$, 8 states naturally correspond to the Cartan of E_8 and those are the states that complete the multiplet. In addition there are 4 more neutral states from the remaining two in $h^{1,1}$ and from H^4 and H^0 . They are in the $\mathbf{1} \oplus \mathbf{3}$ of $SU(2)_1 \subset SO(4)$.

Now we have to address the question of whether the bound state exists at all.

5.1 Bound states in type-IIA

Once we compactify the 11th direction we can study the bound state question as a quantum-mechanical question. On the $0+1$ dimensional intersection of the two 2-branes the massless degrees of freedom are: 4 complex fermions in the $\mathbf{4}$ of $SO(5)$ (the rotations in directions transverse to the two 2-branes) and two complex bosons which are $SO(5)$ scalars but are in the $(+\frac{1}{2}, +\frac{1}{2}) \oplus (-\frac{1}{2}, -\frac{1}{2})$ of $SO(2) \times SO(2)$ (rotations in the 2-brane world-volume keeping the $0+1$ intersection fixed). Those states are oppositely charged under the difference of the two $U(1)$ -s on the two 2-branes and neutral with respect to the sum. The $0+1$ dimensional theory turns out to be the dimensional reduction of $4D \mathcal{N} = 2$ $U(1)$ vector-multiplet coupled to a charged hyper-multiplet. The vector multiplet is the reduction of the difference of the (super) $U(1)$ -s that live on the 2-branes and the hyper-multiplet comes from the fields on the intersection. Since the areas of the 2-branes are finite, the vector-multiplet is also dynamical.

Now we have to take account of the fact that 2 out of the 5 transverse directions are “frozen” because those are directions inside the Calabi-Yau and the 4-cycle \mathcal{S} where all the 2-branes live is isolated. This amounts to decomposing the fields as 4D $\mathcal{N} = 1$ multiplets and making the chiral multiplet which is part of the $\mathcal{N} = 2$ vector multiplet massive. We end up with the dimensional reduction of 4D $\mathcal{N} = 1$ vector multiplet coupled to two oppositely charged chiral multiplets.

5.2 Is there binding in the 11th direction as well?

So far we know that there is stability with respect to separation in directions $1 \dots 4$ when the 11th dimension x_{10} is compactified. Using the techniques of [8] we can also determine whether the state is stable in the 11th direction or not.

As argued in [8], if a bound state exists then it is possible to find a state with $\frac{1}{2}$ a unit of momentum around the 11th direction for each 2-brane. If the bound state did not exist the minimal momentum around the 11th direction for each 2-brane would be 1 unit.

The momentum of a 2-brane around the 11th direction is

$$\int_{\text{2-brane}} \dot{\phi} \quad (32)$$

where ϕ is the coordinate along the 11th direction, which is dual on the 2-brane world-volume to the gauge field A_μ . Thus the momentum is given by

$$\int_{\text{2-brane}} F_{12} \quad (33)$$

where F_{12} is the magnetic field on the 2-brane. So we are looking for a state with half the Dirac unit of magnetic flux on each 2-brane. Stated differently, the two 2-branes have a $U(1)$ holonomy of only π around the origin (where the other 2-brane intersects it).

What is the physical manifestation of that?

We recall that the $0 + 1$ dimensional scalars coming from the DD states connecting the two 2-branes had spinor quantum numbers under rotations in the 89 direction or the 67 direction (i.e. on the 2-brane world-areas). In addition, those scalars were charged one unit under the $U(1)$ of each 2-brane. This means that the field changes sign under a 2π rotation in either of the 2-branes. This is no problem if the wave-function Ψ is even in those scalar

variables, but odd wave-functions can only be defined if there is a $U(1)$ holonomy of π in each 2-brane, so as to make the field single-valued.

Thus, to determine whether there is binding in the 11th dimension as well we must find a wave-function that is normalizable and *odd* with respect to the scalar variables.

We will not go on and explicitly solve the QM problem but we believe that such a state exists.

6 Physical interpretation

Now we can compare the states corresponding to irreducible curves and bound states of curves to the CFT that lives on the non-critical string. Taking $n = 1$ in (11) we have BPS states of the string which wind once. The full Fock space is of the form

$$\prod_l \tilde{b}_{-n_l'}^{A_l \alpha_l} \prod_k \tilde{a}_{-n_k'}^{\nu_k} \prod_i a_{-n_i}^{\mu_i} \prod_j a_{-m_j}^{I_j} |p_L, \theta\rangle \quad (34)$$

$$\mu_i = 1 \dots 4, \quad A_i = 1, 2, \quad I_i = 1 \dots 8 \quad (35)$$

The bosonic oscillators a_{-m}^μ and \tilde{a}_{-m}^μ corresponding to transverse oscillations have indices in the $\underline{4} = (\underline{2}, \underline{2})$ of $SO(4) = SU(2)_1 \times SU(2)_2$ while the fermionic operators $\tilde{b}_{-m}^{A\alpha}$ are in $2(\underline{2}, 0)$. a_{-m}^μ are left-moving while \tilde{a}_{-m}^μ and $\tilde{b}_{-m}^{A\alpha}$ are right movers. The a_{-m}^I oscillators are left-moving internal E_8 lattice oscillators.

The states are naturally in the analog of the Green-Schwarz formalism, but we stress that we have assumed *nothing* about the microscopic description of the string. All we used is the knowledge of the low-energy fields that live on the string.

The Green-Schwarz vacuum $|p_L, \theta\rangle$ is in the $\underline{3} + \underline{1} + 2(\underline{2})$ of $SU(2)_1$ where θ specifies the $SO(4)$ state and p_L specifies the point on the (dual of the) internal E_8 lattice.

BPS states have only left-moving excitations.

$$\prod_i a_{-n_i}^{\mu_i} \prod_j a_{-m_j}^{I_j} |p_L, \theta\rangle \quad (36)$$

Relating (36) to (11) we find

$$\begin{aligned} m &= (L_0 - \bar{L}_0) = L_0 = \sum n_i + \sum m_j \\ \vec{l} &= p_L \end{aligned}$$

The spin of the state in the 4 transverse directions corresponds to the $SO(4)$ representation of (36) which is deduced from the μ_i -s. In the geometrical picture we quoted from [3] that the $SO(4) = SU(2)_1 \times SU(2)_2$ quantum numbers are calculated from the Lefschetz decomposition of $H^*(\mathcal{M})$ and in particular the $SU(2)_2$ spin is at most $\frac{1}{2}$ and the $SU(2)_1$ spin is at most $\frac{1}{2}\dim_{\mathbb{C}}\mathcal{M}$. The $SU(2)_2$ spin states are related to each other by supersymmetry. We will see shortly that this procedure has to be modified because of the special features of our moduli space.

For $m = 0$ there was one state b corresponding to $|0\rangle$. For $m = 1$ we found 2×120 states from 2-branes wrapped on irreducible curves. They were supplemented with 8 more scalar bound states of b and f to form the **248** of E_8 . Those states correspond to $|p_L^2 = -2\rangle$ and $a_{-1}^I|0\rangle$. There were 4 additional states from $H^*(\mathcal{M})$ of the bound state. Those have to correspond to $a_{-1}^\mu|0\rangle$ and thus be in the **(2, 2)** of $SO(4)$ but according to the Lefschetz decomposition they would have to be in **(1, 1) \oplus (3, 1)** so at first sight we seem to have a discrepancy.

What is the difference between our situation and that of [3]?

In our case the moduli space is that of the bound state of b and f . As we saw in section (5), the fermions that live on the 0+1 dimensional intersection of b and f are in a non-trivial representation of $SO(4)$. Thus they modify the $SO(4)$ quantum numbers of the states and eventually we will obtain **(2, 2)** as it should be.

Moving on to higher values of m the geometrical states are either irreducible curves with E_8 representations satisfying (26) or a bound state of an irreducible curve with $0 \leq m' < m$ and $t = m - m'$ 2-branes wrapping the fiber f . The moduli space of such a configuration is

$$\mathcal{M} = \mathcal{S}^t / S_t \tag{37}$$

where each \mathcal{S} comes from one fiber and S_t is the permutation group, since the 2-branes are indistinguishable. The corresponding states will come from $H^*(\mathcal{S}^t / S_t)$ for each curve corresponding to m' .

The construction of the cohomology $H^*(\mathcal{S}^t / S_t)$ was described in [22]. One has to construct a Fock space from operators c_{-n}^w where w corresponds to the states of $H^*(\mathcal{S})$. Thus, for $w = e_1 \dots e_8$ we get 8 bosons which comprise the “internal” E_8 lattice and would be in 1-to-1 correspondence with the a_{-n}^I above. The other 4 values of w are $w = b, f$ and the two

extra states H^0 and H^4 . Those states are in the $\mathbf{\underline{3}} + \mathbf{\underline{1}}$ of $SU(2)_1 \subset SO(4)$ but have to be modified to $(\mathbf{\underline{2}}, \mathbf{\underline{2}})$ because of the fermions at the intersection of the fiber and the base. They will become the a_{-n}^μ of (36). The irreducible curve with “momentum” m' corresponds to the point \vec{p}_L in the dual of the internal E_8 lattice. Thus it seems that the geometrical construction indeed reproduces the Fock space. We have made the assumption that for each p_L there corresponds exactly one irreducible curve. In principle, checking this assumption amounts to counting the number of solutions to polynomial equations in the coefficients of P, Q, R in (21). (A straightforward count of the degrees of the resulting equations, however, didn't seem to work because the equations turned out to have multiplicities. It is probably easier to count the curves in another representation of \mathcal{S} , e.g. a blow-up of \mathbf{CP}^2 . It is also interesting to check whether the reducible curves become irreducible after a deformation away from the elliptic fibration.)

Next come the double covers with $n = 2$. We will consider only the states in the class $2b + 2f$. Their quantum numbers agree with the interpretation as bound states of two singly-wound strings or as a single doubly-wound string. In F-theory a doubly-wound non-critical string has an enhanced $U(2)$ gauge symmetry on it [9] and the 16 fermions that generate the E_8 are in the $\mathbf{\underline{2}}$ of $U(2)$. In 6D we do not expect a bound state (this is $U(2)$ with $\mathcal{N} = 4$ supersymmetry). However, after compactification we can, as in [1], put an appropriate $SO(16) \times SO(16)$ Wilson line and T-dualize to the $SO(32)$ small instanton of [18]. In that theory we are in the Coulomb phase of $Sp(1)$ [1] and there are the $Sp(1)$ gauge bosons whose charge corresponds to $n = 2$ in (11).

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A On the geometry of \mathcal{S}

Starting with the elliptic fibration

$$y^2 = x^3 - f_4(\lambda)x - g_6(\lambda) \quad (38)$$

Which describes a complex surface \mathcal{S} fibered over the exceptional divisor. Let

$$\Delta(\lambda) = \frac{f^3}{27} - \frac{g^2}{4} = \prod_{i=1}^{12} (\lambda - e_i) \quad (39)$$

We assume that ∞ is not a root. We want to describe the 10 2-cycles of this surface. 2 of them are the base and fiber. The remaining form an E_8 intersection matrix and project to paths from e_i to e_j on the base. The condition is that we can find a 1-cycle on the fiber which degenerates at e_i and as we continuously move along the path γ_{ij} from e_i to e_j , we end up again with a degenerate cycle at e_j . All the 1-cycles will then form a 2-cycle with the topology of a sphere.

Let e_i be the vertices of a 12-gon. Let the region inside the 12-gon be R . We can define

$$\sqrt{\Delta} = \sqrt{\prod_{i=1}^{12} (\lambda - \alpha_i)} \quad (40)$$

to be single-valued inside R . The roots of the cubic

$$x^3 - f_4(\lambda)x - g_6(\lambda) = \prod_{k=1}^3 (\lambda - \alpha_k) \quad (41)$$

are given by

$$\alpha_k = e^{\frac{2\pi i k}{3}} \left(\frac{g}{2} + \sqrt{\Delta} \right)^{1/3} + \frac{f/3}{e^{\frac{2\pi i k}{3}} \left(\frac{g}{2} + \sqrt{\Delta} \right)^{1/3}} \quad (42)$$

When $\Delta = 0$, $\alpha_1 = \alpha_2$. Let us draw cuts for $\sqrt{\Delta}$ from α_{2k-1} to α_{2k} . Then there are \mathbf{Z}_2 monodromies around each α_{2k} but as we shall soon see, there are also \mathbf{Z}_3 monodromies (in the order $\alpha_1, \alpha_2, \alpha_3$) when making a loop around each of the cuts. In fact, $\sqrt{\Delta}$ is defined on a double cover which is a Riemann surface of genus 5. The monodromies are around the non-trivial 1-cycles. The cubic has 3 roots for each z . When going around cycles around the α_k -s the 3 roots get interchanged. The monodromies are thus in S_3 . The roots can be ordered so that the monodromies for loops that start at the origin and go around the α_k -s are as follows:

$$\text{Monodromy} = \begin{cases} (12) & \text{for a loop around } \alpha_{2k+1} \\ (13) & \text{for a loop around } \alpha_{2k+2} \end{cases} \quad (43)$$

Let us find 8 2-cycles whose intersection matrix is E_8 . To find the 2-cycles in $H^2(S, \mathbf{Z})$ we must pick paths from α_k to α_j in such a way that when we start with a 1-cycle in the torus that shrinks at α_k and see what happens to the cycle as we move along the path γ to α_j then it has to shrink again as we reach α_k . Every 2-cycle as above has a self-intersection of (-2) . Cycles that go from α_1 to α_{2k+1} (like α in Fig C) from the outside, have monodromy:

$$(12)(123)^k(12)(123)^k \quad (44)$$

and this is (1) only if $k = 3$. Cycles that go from α_1 to α_{2k} from the outside have monodromy

$$(123) \quad (45)$$

and are never trivial. Cycles that go from α_2 to α_{2k+1} have also monodromy (123) and are never trivial.

Now we can give a basis for $H^2(\mathcal{S}, \mathbf{Z})$. We take the 8 2-cycles: (1) from α_1 to α_3 , (2) from α_2 to α_{12} , (3) from α_{12} to α_4 , (4) from α_4 to α_6 , (5) from α_5 to α_7 , (6) from α_7 to α_9 , (7) from α_8 to α_{10} and (8) from α_5 to α_{11} but the last one is from the outside.

Their intersection forms the E_8 matrix.

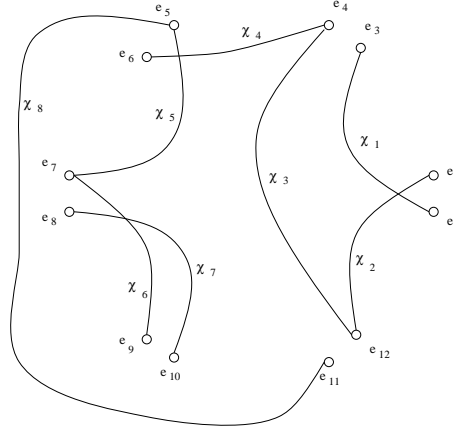


Fig.A: The 2-cycles as paths

B Polynomial sections

We want to prove that there are no polynomials $y(\lambda)$ and $x(\lambda)$ such that

$$\deg(x(\lambda)) > 8 \quad (46)$$

and

$$y^2 = x^3 - f_4x - g_6 \quad (47)$$

Let $e_i(\lambda)$ be the three roots of the cubic. The functions $e_i(\lambda)$ give rise to a single valued function $e(z)$ that lives on a triple cover of the λ -plane which is of genus 4:

$$2g - 2 = 3 \cdot (-2) + 12 = 6 \quad (48)$$

and there are 12 simple branch-points. Let's call this Riemann surface Σ_4 . Now

$$y^2 = (x - e_1)(x - e_2)(x - e_3) \quad (49)$$

Each zero of the RHS is at least a double zero. Now consider the function

$$\mu_i = \sqrt{x - e_i} \quad (50)$$

Either each zero of $x - e_i$ is a double zero, in which case μ_i corresponds to a single-valued function $\mu(z)$ on Σ_4 or $x - e_i$ and $x - e_j$ have a common root. This can only happen when $e_i = e_j$ i.e. at one of the 12 branch points of the cover $\Sigma_4 \rightarrow \mathbf{CP}^1$.

Let us start with the first case.

$$x(\lambda(z)) = \mu(z)^2 - e(z) \quad (51)$$

where $\lambda(z)$ is a rational function on Σ_4 with three simple zeroes (and hence three simple poles) and with 12 simple zeroes of its derivative. μ and e and $x(\lambda)$ are rational functions on Σ_4 . Let us count the number of poles of the LHS and RHS. The poles can be at one of the 3 points $\lambda^{-1}(\infty)$. $e(z)$ has 6 simple zeroes (zeroes of g_6) and thus three double poles. Let n be the degree of $x(\lambda)$. Then $x(\lambda)$ has three poles of multiplicity n . We see that for $n > 2$ $\mu(z)^2$ also has three poles of multiplicity n . Thus $n = 2m$ is even and μ has three poles of multiplicity m .

Now we proceed as follows. For a generic $\lambda_0 \in \mathbf{CP}^1$ there are three values of z_i with $i = 1, 2, 3$ for which $\lambda(z_i) = \lambda_0$. For non-generic λ_0 we take appropriate multiplicities. Let

$$\begin{aligned}\tilde{\Delta}(\lambda_0) &= \prod_{1 \leq i < j \leq 3} (\mu(z_i) - \mu(z_j))^2 \\ \Xi(\lambda_0) &= \prod_{1 \leq i < j \leq 3} (\mu(z_i) + \mu(z_j))\end{aligned}$$

$\tilde{\Delta}$ and Ξ are single valued functions of λ_0 and thus are polynomials. Furthermore

$$\tilde{\Delta}\Xi^2 = \prod_{i < j} (\mu(z_i)^2 - \mu(z_j)^2)^2 = \prod_{i < j} (e(z_i) - e(z_j))^2 = \Delta_{12} \quad (52)$$

where we used (51) and Δ is the discriminant

$$\Delta_{12}(\lambda) = \frac{f^3}{27} - \frac{g^2}{4} \quad (53)$$

Let us calculate the degrees of the polynomials Ξ and $\tilde{\Delta}$. Near $\lambda = \infty$ and for $n > 2$

$$\mu_i = \pm \sqrt{x(\lambda) + e_i} \sim \pm \lambda^m \quad (54)$$

The \pm signs for the three \pm_i -s can either all be the same or be two of one kind and the third of the opposite sign. In the first case

$$\Xi = \prod_{i < j} (\mu(z_i) + \mu(z_j)) \sim \lambda^{3m} \quad (55)$$

but then Ξ^2 has degree $6m$ which is greater than 12 for $m > 2$ and contradicts (52). for $m = 2$ we find

$$\tilde{\Delta} = \text{const} \quad (56)$$

we will soon discuss this case as well, but let us first see what happens when the signs in (54) are different, say two (+)'s for μ_1 and μ_2 and one (−) for μ_3 . Then

$$\begin{aligned}\mu_1 + \mu_2 &= \sqrt{x + e_1} + \sqrt{x + e_2} \sim \lambda^m \\ \mu_1 + \mu_3 &= \sqrt{x + e_1} - \sqrt{x + e_3} = \frac{e_1 - e_3}{\sqrt{x + e_1} + \sqrt{x + e_3}} \sim \lambda^{2-m} \\ \mu_2 + \mu_3 &= \sqrt{x + e_2} - \sqrt{x + e_3} \sim \lambda^{2-m} \\ \Xi &\sim \lambda^{4-m}\end{aligned}$$

we see that (52) requires $m \leq 4$. The remaining cases have

$$\deg(\Xi) \leq 3 \quad (57)$$

Now we can write define the symmetric sums

$$\begin{aligned}
\sigma_1 &= \sum \mu_i \\
\sigma_2 &= \sum \mu_i^2 = 3x(\lambda) + \sum e_i = 3x(\lambda) \\
\sigma_3 &= \sum \mu_i^3 \\
\sigma_4 &= \sum \mu_i^2 = \sum (x(\lambda) + e_i)^2 = 3x(\lambda)^2 + 2f_4(\lambda)
\end{aligned}$$

some algebra gives:

$$\begin{aligned}
\Xi &= \prod_{i < j} (\mu(z_i) + \mu(z_j)) = \frac{1}{3}(\sigma_1^3 - \sigma_3) \\
0 &= 6\sigma_4 + 6\sigma_2\sigma_1^2 - 3\sigma_2^2 - \sigma_1^4 - 8\sigma_1\sigma_3
\end{aligned}$$

so we find

$$(\sigma_1^2 - x)^2 = \frac{4}{3}f_4 - \frac{8}{3}\Xi\sigma_1 \quad (58)$$

The RHS is a polynomial of degree 4 at most. Thus $\sigma_1^2 - x$ is of degree 2 and we find

$$x = p_m^2 + q_2 \quad (59)$$

where q_2 is of degree 2 (at most) and p_m is of degree m (exactly). Putting back in the original equation we find

$$y_{3m}^2 = x_{2m}^3 - f_4 x_{2m} - g_6 = (p_m^3 + \frac{3}{2}p_m q_2)^2 - \frac{9}{4}p_m^2(q_2^2 - \frac{4}{3}q_2 + \frac{4}{9}f_4) + (q_2^3 - f_4 q_2 - g_6) \quad (60)$$

we are looking for non-generic solutions, i.e. solutions for special values of f_4 and g_6 . Now let's separate into cases:

$m = 2$

This case admits solutions for a co-dimension 1 subspace of the parameter space (of coefficients of f_4 and g_6). We will see that generically this subspace of moduli space corresponds to a nonsingular surface.

This case also includes the missing case above where all the signs of (54) are equal.

In this case every x_4 can be written as

$$x = p_2^2 + q_1 \quad (61)$$

After some algebra we find the general solution

$$\begin{aligned} f_4 &= -2p_2r_2 + t_1 \\ g_6 &= -\frac{9}{4}p_2^2q_1^2 - p_2r_2q_1 - p_2^2t_1 - r_2^2 - t_1q_1 \end{aligned}$$

Every choice of p_2 , q_2 , r_2 , q_1 and t_1 (the subscripts are degrees of the polynomials) gives appropriate g_6 and f_4 .

$m = 3$

Similarly we find

$$x = p_3^2 + q_2 \tag{62}$$

and

$$\begin{aligned} f_4 &= -\frac{9}{4}q_2^2 + 3q_2 - v_1p_3 - c_0 \\ g_6 &= \frac{13}{4}q_2^3 - 3q_2^2 + v_1p_3q_2 + c_0q_2 - v_1^2 - p_3q_2v_1 + c_0p_3^2 \end{aligned}$$

$m = 4$

$$x = p_4^2 + q_2 \tag{63}$$

and

$$\begin{aligned} f_4 &= -\frac{9}{4}q_2^2 + 3q_2 - 2c_0p_4 \\ g_6 &= \frac{13}{4}q_2^3 - 3q_2^2 + 2c_0q_2p_4 - 3p_4c_0 - c_0^2 \end{aligned}$$

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